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Properties of Clebsch–Gordan coefficients and 3-j symbols for the quantum superalgebra $U_q(osp(1|2))$

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Abstract. This paper is a continuation of the study begun in an earlier paper of the structure of grade-star representations of the quantum superalgebra $U_q(osp(1|2))$. The general case of the tensor product of two grade-star representations acting in representation spaces with arbitrary (not necessarily positive definite) Hermitian forms is considered. An explicit analytical formula for Clebsch–Gordan coefficients for this general case is derived using the projection operator method. Pseudo-orthogonality relations are given and symmetry properties, including Regge symmetry, are discussed. The quantum analogues of super 3-j symbols are defined and their symmetry properties are analysed.

1. Introduction

Recently, quantum algebras [1] have provoked considerable interest among theoretical physicists. This wide interest may be explained by the fact that quantum algebras are continuous deformations of well known Lie algebras and that their representation theory is very similar to that of non-deformed Lie algebras. In particular, the Hermitian representations of the quantum algebra $U_q(su(2))$ have the same structure as those of the $su(2)$ algebra. It has been shown in several papers [2–5], that for the Hermitian representations of quantum algebra $U_q(su(2))$, the Racah–Wigner calculus can be fully developed following the same lines as in the classical case. It is quite remarkable that all topics that are relevant for the classical Racah–Wigner calculus have their direct quantum analogue in the representation theory of the quantum algebra $U_q(su(2))$.

A very efficient method for the analysis of the properties of irreducible representations is the projection operator method, first introduced to derive the $su(2)$ Clebsch–Gordan coefficients (CGC) by Shapiro [6]. Recently, Smirnov, Tolstoy and Kharitonov [7, 8] have used the projection operator method to derive an analytical formula for the CGC of the quantum algebra $U_q(su(2))$ and to study the corresponding Racah–Wigner calculus.

The superalgebra $osp(1|2)$ was first introduced by Pais and Rittenberg [9]. In [10], Scheunert, Nahm and Rittenberg introduced the concept of grade-star representations which are generalizations of Hermitian representations of simple Lie algebras. In the case of the superalgebra $osp(1|2)$ the grade-star operation is a generalization of the classical star operation in the sense that when it is restricted to the even part $sl(2)$ of $osp(1|2)$, it becomes a classical star operation (which is associated with the real form of $su(2)$). Because real

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forms of $osp(1|2)$ do not contain the Lie algebra $su(2)$ as their even part [11], the grade-star operation is not associated with any real form of $osp(1|2)$, but it allows one to construct superanalogues of the Hermitian representations. The structure of grade-star representations of $osp(1|2)$ is very similar to the structure of Hermitian representations of $su(2)$. In particular, even generators of $osp(1|2)$ are represented in the same way as in Hermitian representations of $su(2)$. The tensor product of two irreducible grade-star representations is simply reducible and using the inclusion $sl(2) \subset osp(1|2)$, Scheunert, Nahm and Rittenberg [12] and Berezin and Tolstoy [13] showed that any $osp(1|2)$ Clebsch–Gordan coefficient can be factorized into the product of a usual $su(2)$ Clebsch–Gordan coefficient and a so-called scalar factor. In [14–16] (see also references therein), it has been shown that the Racah–Wigner calculus can also be constructed for this superalgebra. In particular, the super $s3$-j and super $s6$-j symbols have been defined and expressed in terms of the classical 3-j and 6-j symbols.

The quantum superalgebra $U_q(osp(1|2))$, which is the subject of this paper, is the quantum analogue of the $osp(1|2)$ superalgebra. The grade-star representations of $U_q(osp(1|2))$ are superanalogues of Hermitian representations of the $U_q(su(2))$ quantum algebra. It should be noted that there is no inclusion $U_q(sl(2)) \not\subset U_q osp(1|2))$. This quantum superalgebra has been defined and studied by Kulish and Reshetikhin [17, 18]. Its Clebsch–Gordan coefficients were derived by Kulish [19], using a recursion relation, and in [20] using the projection operator method. A particular case of CGC was also given by Saleur [21].

In [20], it was shown that in the reduction of the tensor product of two irreducible representation spaces of $U_q(osp(1|2))$ with positive definite bilinear Hermitian forms, representation spaces appear, the Hermitian forms of which are not positive definite and where the highest-weight vector is normalized to -1. In this paper, in order to study the most general case, we consider the reduction of the tensor product of representation spaces whose bilinear Hermitian forms are not necessarily positive definite, and using the projection operator method, we derive an analytical formula for the Clebsch–Gordan coefficients. This analytical formula does not differ from the analytical formula obtained in [20], which proves that the Clebsch–Gordan coefficients do not depend on the signatures of the bilinear Hermitian forms defined in the representation spaces. In addition, we study several properties of Clebsch–Gordan coefficients: orthogonality relations, symmetry properties, particular values and also show that Clebsch–Gordan coefficients satisfy a conditional Regge symmetry.

As in the case of $U_q(su(2))$, the study of the symmetry properties of the Clebsch–Gordan coefficients allows one to define $sq3$-j symbols for $U_q(osp(1|2))$ that possess good symmetry properties. We first define $sq3$-$j\lambda$ symbols which depend on the parities λ of the graded representation space bases. Then, we show that the dependence on parities can be factorized out, so that one can define parity-independent 3-j symbols for the quantum superalgebra $U_q(osp(1|2))$ that are superanalogues of the 3-j symbols for the quantum algebra $U_q(su(2))$ and that have symmetry properties very similar to those of these 3-j symbols [7, 8].

This paper has the following structure. Section 2 contains the definition of the quantum superalgebra $U_q(osp(1|2))$ and the basic properties of its irreducible representations, and we recall the explicit expression of the projection operator for the quantum superalgebra $U_q(osp(1|2))$. In section 3, we consider the tensor product of two irreducible representations of $U_q(osp(1|2))$ with arbitrary Hermitian forms and the projection operator is used to derive an analytical formula for the Clebsch–Gordan coefficients. Pseudo-orthogonality relations, recursion relations and symmetry properties of the Clebsch–Gordan coefficients are given in the following subsections of section 3. Section 4 is devoted to 3-j symbols: parity-dependent and parity-independent 3-j symbols for $U_q(osp(1|2))$ are defined and their properties are discussed.

2. The irreducible representations of the quantum superalgebra $U_q(osp(1|2))$

2.1. Representation spaces

The quantum superalgebra $U_q(osp(1|2))$ is generated by three elements: H (even) and $v_\pm$ (odd) with the following (anti)commutation relations

$$[H, v_\pm] = \pm\tfrac{1}{2}v_\pm \qquad [v_+, v_-]_+ = -\frac{\mathrm{sh}(\eta H)}{\mathrm{sh}(2\eta)} \tag{2.1}$$

where the deformation parameter η is real and is related to the q-deformation parameter by $q = \mathrm{e}^{-\frac{\eta}{2}}$. The following expressions for coproduct Δ, antipode S and counit ϵ define on $U_q(osp(1|2))$ the structure of a Hopf algebra

$$\Delta(v_\pm) = v_\pm \otimes q^{H} + q^{-H} \otimes v_\pm \tag{2.2}$$

$$\Delta(H) = H \otimes \mathbf{1} + \mathbf{1} \otimes H \qquad \Delta(\mathbf{1}) = \mathbf{1} \otimes \mathbf{1} \tag{2.3}$$

$$S(H) = -H \qquad S(v_\pm) = -q^{\pm\frac{1}{2}} v_\pm \tag{2.4}$$

$$\epsilon(H) = \epsilon(v_\pm) = 0 \qquad \epsilon(\mathbf{1}) = \mathbf{1}\,. \tag{2.5}$$

For the homogenous elements of $U_q(osp(1|2))$, one can define a parity function deg such that

$$\deg(H) = 0 \qquad \deg(v_\pm) = 1 \qquad \deg(\mathbf{1}) = 0\,. \tag{2.6}$$

A finite-dimensional representation space V of $U_q(osp(1|2))$ is a graded vector space $V = V(0) \oplus V(1)$ where $V(0)$ is an even subspace and $V(1)$ is an odd subspace. We assume that there exists in V a bilinear Hermitian form (,), not necessarily positive definite, such that

$$(V(0), V(1)) = 0\,. \tag{2.7}$$

A representation of the quantum superalgebra $U_q(osp(1|2))$ in the finite-dimensional graded space V is a homomorphism T

$$T : U_q(osp(1|2)) \to L(V, V) \tag{2.8}$$

of the associative graded algebra $U_q(osp(1|2))$ in the associative graded algebra of linear operators in V, $L(V, V)$, such that

$$[T(H), T(v_\pm)] = \pm\tfrac{1}{2}T(v_\pm) \qquad [T(v_+), T(v_-)]_+ = -\frac{\mathrm{sh}(\eta T(H))}{\mathrm{sh}(2\eta)}\,. \tag{2.9}$$

From the (anti)commutation relations (2.1), one can derive the following fundamental formula:

$$(T(v_+))^m (T(v_-))^n = \sum_{i=0}^{\min(m,n)} (-1)^{mn} (-1)^{\frac{i(i-1)}{2}} \frac{[m]![n]!}{[i]![m-i]![n-i]!}$$
$$\times (T(v_-))^{n-i} (T(v_+))^{m-i} \frac{[4T(H) - n + m]!}{[4T(H) - n + m - i]!} \gamma^i \tag{2.10}$$

where

$$\gamma = \frac{\mathrm{ch}(\frac{\eta}{4})}{\mathrm{sh}(2\eta)} \tag{2.11}$$

and $[n]$ is the Kulish symbol defined as follows [19]

$$[n] = \frac{q^{-\frac{n}{2}} - (-1)^n q^{\frac{n}{2}}}{q^{-\frac{1}{2}} + q^{\frac{1}{2}}} = \begin{cases} \dfrac{\mathrm{sh}(\frac{\eta}{4}(n))}{\mathrm{ch}(\frac{\eta}{4})} & \text{if } n \text{ is even} \\ \\ \dfrac{\mathrm{ch}(\frac{\eta}{4}(n))}{\mathrm{ch}(\frac{\eta}{4})} & \text{if } n \text{ is odd.} \end{cases} \tag{2.12}$$

We have set $q = \mathrm{e}^{-\frac{\eta}{2}}$ therefore, the symbol $[n]$ is positive if $\eta > 0$. Note that the limit $q \to 1$ depends on the parity of the argument n of the symbol. Scheunert, Nahm and Rittenberg [10] have introduced the concept of grade-star representations. In such a representation the operators satisfy the following relations:

$$T(H)^\star = T(H) \qquad T(v_\pm)^\star = \pm(-1)^\epsilon T(v_\mp) \qquad T(1)^\star = T(1) \tag{2.13}$$

where $(\star)$ is the grade adjoint operation defined in the following way

$$(T(X)^\star f, g) = (-1)^{\deg(X)\deg(f)}(f, T(X)g) \tag{2.14}$$

for any $X \in U_q(osp(1|2))$ and $f, g \in V$. The index $\epsilon = 0, 1$ defines the class of the representation.

For more details on grade-star representations see [12, 13, 16, 20].

2.2. *Finite-dimensional irreducible representations*

Let v^l be a finite-dimensional rapresentation space with highest-weight l (l is a non-negative integer). The highest-weight vector is denoted by e^l_l and is defined by the following properties:

$$T(H)(e^l_l) = \frac{l}{2} e^l_l \qquad T(v_+)(e^l_l) = 0 \tag{2.15}$$

$$(e^l_l, e^l_l) = (-1)^\psi \qquad \text{with } \psi = 0, 1. \tag{2.16}$$

The last condition is motivated by the fact that, for a tensor product of two irreducible representations of $U_q(osp(1|2))$ with positive definite bilinear Hermitian forms, in the Clebsch–Gordan series representation spaces appear whose Hermitian forms are not positive definite and where the highest-weight vector is normalized to -1 [20]. Therefore, in order to study the general case, we consider representation spaces where the condition (2.16) holds.

From the relations (2.15), it follows that e^l_l belongs either to $V^l(0)$ or to $V^l(1)$, i.e. it has a definite parity. Therefore, we set $e^l_l \equiv e^l_l(\lambda)$ and $V^l \equiv V^l(\lambda)$, where $\lambda = 0, 1$ is the parity of the highest-weight vector in the graded representation space.

Thus, any grade-star representation of $U_q(osp(1|2))$ is characterized by four parameters: the superspin l (a non-negative integer), the parity $\lambda = 0, 1$, the normalization parameter $\psi = 0, 1$ and the class $\epsilon = 0, 1$ of the representation.

One can construct in the usual way an orthogonal basis $e^{lq}_m(\lambda)$ in $V^l(\lambda)$ where $m = l, l-1, \ldots -l+1, -l$, so that the representation space $V(\lambda)$ is $(2l+1)$-dimensional.

The vectors $e^{lq}_m(\lambda)$ are orthogonal and normalized to ± 1; more precisely we have

$$(e^{lq}_m(\lambda), e^{lq}_{m'}(\lambda)) = (-1)^{\varphi(l-m)+\psi} \delta_{mm'} \tag{2.17}$$

where

$$\varphi = \lambda + \epsilon + 1 \quad \mathrm{mod}(2)\,. \tag{2.18}$$

In the particular case

$$\psi = 0 \qquad \varphi = 0 \Leftrightarrow \lambda = \epsilon + 1 \quad \mathrm{mod}(2) \tag{2.19}$$

the basis vectors $e_m^{lq}(\lambda)$ are normalized to +1, which means that the Hermitian form (,) is positive definite. This case was considered in [20]. In the following, we shall consider the general case where φ and ψ are not fixed.

The operators $T(v_\pm)$ and $T(H)$ act on the basis $e_m^{lq}(\lambda)$ in the following way:

$$T(H)e_m^{lq}(\lambda) = \frac{m}{2}\, e_m^{lq}(\lambda) \tag{2.20}$$

$$T(v_+)e_m^{lq}(\lambda) = (-1)^{l-m}\sqrt{[l-m][l+m+1]\gamma}\, e_{m+1}^{lq}(\lambda) \tag{2.21}$$

$$T(v_-)e_m^{lq}(\lambda) = \sqrt{[l+m][l-m+1]\gamma}\, e_{m-1}^{lq}(\lambda)\,. \tag{2.22}$$

This action of the operators $T(v_\pm)$ and $T(H)$ does not depend on the parameters λ, ϵ, ψ.

The grade-star representation T, characterized by the class ϵ and acting in the representation space $V^l(\lambda)$ whose Hermitian form signature is determined by φ and ψ is denoted by $T^{l\epsilon}_{\varphi\psi}$ (the index λ is fixed by relation (2.18)). However, in the following, the indices ϵ, φ, ψ will often be omitted in the notation.

In the limit $q \to 1$, we obtain a basis for the superalgebra $osp(1|2)$ representation space, cf [20]

$$\lim_{q\to 1} e_m^{lq}(\lambda) = e_m^l(\lambda)\,. \tag{2.23}$$

2.3. The projection operator P^q for the quantum superalgebra $U_q(osp(1/2))$

In this subsection, we recall the definition and some properties of the projection operator for the quantum superalgebra $U_q(osp(1|2))$. This operator P^q acts linearly in the space V, the direct sum of all representation spaces V^l. It is defined by the following requirements:

$$[T(H), P^q] = 0 \qquad T(v_+)P^q = 0 \qquad (P^q)^2 = P^q \qquad P^q\, e_l^{lq}(\lambda) = e_l^{lq}(\lambda)\,. \tag{2.24}$$

It has been shown in [20], that the operator P^q can be written in the form of a series

$$P^q = \sum_{r=0}^{\infty} c_r(T(H))(T(v_-))^r(T(v_+))^r \tag{2.25}$$

where

$$c_r(T(H)) = \frac{[4T(H)+1]!}{[4T(H)+r+1]![r]!\gamma^r}\,. \tag{2.26}$$

General formulae for the projection operator of quantum orthosymplectic superalgebras have been derived by Koroshkin and Tolstoy [22]. In the limit $q \to 1$, this coefficient and therefore the projection operator are equal to the corresponding $osp(1|2)$ coefficient and projection operator P, cf [13],

$$\lim_{q\to 1} P^q = P\,. \tag{2.27}$$

Let us consider the space W_m of all vectors of weight m, i.e. $W_m = \{f | T(H)f = \frac{m}{2}f\}$. The restriction of P^q to this space is denoted by P^{mq} and it has the form

$$P^{mq} = \sum_{r=0}^{\infty} c_r(m)(T(v_-))^r(T(v_+))^r \tag{2.28}$$

where the coefficients $c_r(m)$ are now numbers

$$c_r(m) = \frac{[2m+1]!}{[2m+r+1]![r]!\gamma^r}\,. \tag{2.29}$$

In what follows, we will use the so-called shift operators P^{lq}_{mn}, acting in V_l. For $l \geqslant m$ and $l \geqslant n$ they are defined by the expression

$$P^{lq}_{mn} = (-1)^{\frac{1}{2}(l-n)(l-n+1)} \sqrt{\frac{[l+m]!}{[2l]![l-m]!}\gamma^{-(l-m)}} \sqrt{\frac{[l+n]!}{[2l]![l-n]!}\gamma^{-(l-n)}}$$

$$\times (T(v_-))^{l-m}\, P^{lq}\, (T(v_+))^{l-n}\,. \tag{2.30}$$

For properties of the shift operators P^{lq}_{mn} see [20].

3. The Clebsch–Gordan coefficients for the quantum superalgebra $U_q(osp(1|2))$

3.1. Tensor product of two irreducible representations

Let $V^{l_1}(\lambda_1)$ and $V^{l_2}(\lambda_2)$ be the representation spaces of two representations $T^{l_1\epsilon}_{\varphi_1\psi_1}$ and $T^{l_2\epsilon}_{\varphi_2\psi_2}$ of the same class ϵ. From (2.18), this implies that the parities λ_i and signatures φ_i $(i = 1, 2)$ are related by

$$\lambda_1 + \varphi_1 = \lambda_2 + \varphi_2 \quad \mathrm{mod}(2)\,. \tag{3.1}$$

The bilinear Hermitian form in the tensor product space $V^{l_1}(\lambda_1) \otimes V^{l_2}(\lambda_2)$ is defined by

$$((X_1 \otimes X_2)\,,\,(Y_1 \otimes Y_2)) = (-1)^{\deg(X_2)\deg(Y_1)}(X_1, Y_1)(X_2, Y_2) \tag{3.2}$$

where $X_1, Y_1 \in V^{l_1}(\lambda_1)$, $X_2, Y_2 \in V^{l_2}(\lambda_2)$. It should be stressed that even if both Hermitian forms in the representation spaces $V^{l_1}(\lambda_1)$ and $V^{l_2}(\lambda_2)$ are positive definite, the form (3.2) is not necessarily positive definite.

The space $V^{l_1}(\lambda_1) \otimes V^{l_2}(\lambda_2)$ with basis vectors $e^{l_1q}_{m_1}(\lambda_1) \otimes e^{l_2q}_{m_2}(\lambda_2)$ is a representation space for the tensor product of representations $T^{l_1\epsilon}_{\varphi_1\psi_1} \otimes T^{l_2\epsilon}_{\varphi_2\psi_2}$ and the action of the generators $v_\pm$ and H in this space is represented by the following operators:

$$v^{\otimes}_{\pm}(1,2) = (T^{l_1} \otimes T^{l_2})\Delta(v_\pm) = T^{l_1}(v_\pm) \otimes q^{T^{l_2}(H)} + q^{-T^{l_1}(H)} \otimes T^{l_2}(v_\pm) \tag{3.3}$$

$$H^{\otimes}(1,2) = (T^{l_1} \otimes T^{l_2})\Delta(H) = T^{l_1}(H) \otimes T^{l_2}(\mathbb{1}) + T^{l_1}(\mathbb{1}) \otimes T^{l_2}(H)\,. \tag{3.4}$$

The grade adjoints of operators $v^{\otimes}_{\pm}(1,2)$ and $H^{\otimes}(1,2)$ are defined by

$$(v^{\otimes}_{\pm}(1,2))^\star = (T^{l_1}(v_\pm))^\star \otimes q^{(T^{l_2}(H))^\star} + q^{-(T^{l_1}(H))^\star} \otimes (T^{l_2}(v_\pm))^\star \tag{3.5}$$

$$(H^{\otimes}(1,2))^\star = (T^{l_1}(H))^\star \otimes T^{l_2}(\mathbb{1})^\star + T^{l_1}(\mathbb{1})^\star \otimes (T^{l_2}(H))^\star \tag{3.6}$$

i.e. the grade adjoint operation does not change the order in the coproduct. With this definition, the tensor product of the representations $T^{l_1\epsilon}_{\varphi_1\psi_1} \otimes T^{l_2\epsilon}_{\varphi_2\psi_2}$ is a representation of the class ϵ with respect to the Hermitian form (3.2).

It has been shown in [20] that the tensor product of the representation spaces $V^{l_1}(\lambda_1) \otimes V^{l_2}(\lambda_2)$ can be reduced to a direct sum of the subspaces $V^l(\lambda)$

$$V^{l_1}(\lambda_1) \otimes V^{l_2}(\lambda_2) = \oplus_l V^l(\lambda) \tag{3.7}$$

where l is an integer satisfying the conditions

$$|l_1 - l_2| \leqslant l \leqslant l_1 + l_2\,. \tag{3.8}$$

In this reduction, each subspace appears only once, i.e. the tensor product of two representations of the same class is simply reducible.

3.2. $U_q((1/2))$ Clebsch–Gordan coefficients

By definition, the Clebsch–Gordan coefficients $(l_1m_1\lambda_1, l_2m_2\lambda_2|lm\lambda)_q$ relate the pseudo-normalized basis $e^{l_1q}_{m_1}(\lambda_1) \otimes e^{l_2q}_{m_2}(\lambda_2)$ and the reduced pseudo-normalized basis $e^{lq}_m(l_1, l_2, \lambda)$ in the following way:

$$e^{lq}_m(l_1, l_2, \lambda) = \sum_{m_1,m_2} (l_1m_1\lambda_1, l_2m_2\lambda_2|lm\lambda)_q\, e^{l_1q}_{m_1}(\lambda_1) \otimes e^{l_2q}_{m_2}(\lambda_2) \tag{3.9}$$

where $m_1 + m_2 = m$. From the definition (3.2) of the bilinear form in $V^{l_1}(\lambda_1) \otimes V^{l_2}(\lambda_2)$ it follows that:

$$\begin{aligned}(l_1m_1\lambda_1, l_2m_2\lambda_2|lm\lambda)_q &= (-1)^{(l_1-m_1+\lambda_1)(l_2-m_2+\lambda_2)}\,(-1)^{\left(\sum_{i=1}^{2}\varphi_i(l_i-m_i)+\psi_i\right)}\\ &\quad\times(e^{l_1q}_{m_1}(\lambda_1) \otimes e^{l_2q}_{m_2}(\lambda_2), e^{lq}_m(l_1, l_2, \lambda)) .\end{aligned} \tag{3.10}$$

The vector $e^{lq}_m(l_1, l_2, \lambda)$ is represented without loss of generality [20] in the form

$$e^{lq}_m(l_1, l_2, \lambda) = \frac{1}{\sqrt{|N(q, l, m)|}} P^{lq\otimes}_{ml}(e^{l_1q}_{l_1}(\lambda_1) \otimes e^{l_2q}_{l-l_1}(\lambda_2)) . \tag{3.11}$$

Using the properties of the shift operators P^{lq}_{ml} the normalization factor $N(q, l, m)$ is shown to be

$$\begin{aligned}N(q, l, m) &= (P^{lq\otimes}_{ml}(e^{l_1q}_{l_1}(\lambda_1) \otimes e^{l_2q}_{l-l_1}(\lambda_2)), P^{lq\otimes}_{ml}(e^{l_1q}_{l_1}(\lambda_1) \otimes e^{l_2q}_{l-l_1}(\lambda_2)))\\ &= (-1)^{\varphi(l-m)+\psi}\, q^{\frac{1}{2}(l_1+l_2-l)(l+l_2-l_1+1)} \frac{[2l + 1]![2l_1]!}{[l_1 - l_2 + l]![l_1 + l_2 + l + 1]!}\end{aligned} \tag{3.12}$$

and therefore, one has

$$\begin{aligned}&(e^{lq}_m(l_1, l_2, \lambda), e^{l'q}_{m'}(l_1, l_2, \lambda))\\ &= \frac{1}{|N(q, l, m)|}(P^{lq\otimes}_{ml}(e^{l_1q}_{l_1}(\lambda_1) \otimes e^{l_2q}_{l-l_1}(\lambda_2)) ,\ P^{l'q\otimes}_{m'l'}(e^{l_1q}_{l_1}(\lambda_1) \otimes e^{l_2q}_{l'-l_1}(\lambda_1)))\\ &= \frac{N(q, l, m)}{|N(q, l, m)|}\delta_{ll'}\,\delta_{mm'} = \delta_{ll'}\,\delta_{mm'}\,(-1)^{\varphi(l-m)+\psi}\end{aligned} \tag{3.13}$$

where

$$\varphi = l_1 + l_2 + l + \lambda_1 + \varphi_2 \quad \mathrm{mod}(2) \tag{3.14}$$

$$\psi = (l_1 + l_2 + l + \lambda_2)\lambda_1 + \varphi_2(l_1 + l_2 + l) + \psi_1 + \psi_2 \quad \mathrm{mod}(2) . \tag{3.15}$$

Thus, the basis $e^{lq}_m(l_1, l_2, \lambda)$ is orthogonal but not positive definite. Its signature is the same as in the classical $osp(1|2)$ case [16]. If we set $\varphi_i = \psi_i = 0,\ i = 1, 2$, we obtain the same formula as in [20].

From the relation (3.11), it follows immediately that the parity λ of the reduced basis $e^{lq}_m(l_1, l_2, \lambda)$ can be expressed as

$$\lambda = l_1 + l_2 + l + \lambda_1 + \lambda_2 \quad \mathrm{mod}(2) . \tag{3.16}$$

Using the properties of the projection operators, and after laborious calculation, we get the following analytical expression for the Clebsch–Gordan coefficients of the quantum superalgebra $U_q(osp(1|2)$:

$$(l_1 m_1 \lambda_1, l_2 m_2 \lambda_2 | l m \lambda)_q =$$

$$(-1)^{\lambda_1(l+l_1-m_2)} (-1)^{\frac{1}{2}(l_1+l_2-l)(l_1+l_2-l+1)} q^{-\frac{1}{4}(l_1+l_2-l)(l+l_2+l_1+1)+\frac{1}{2}l_1 m_2 - l_2 m_1}$$

$$\times \left([2l+1] \frac{[l_1-l_2+l]![l+m]![l+l_2+l_1+1]![l_2-m_2]![l_1+l_2-l]!}{[l-m]![l_1+m_1]![l_2+m_2]![l_2-l_1+l]![l_1-m_1]!} \right)^{\frac{1}{2}}$$

$$\times \sum_z (-1)^{\frac{1}{2}z(z+1)} q^{\frac{1}{2}z(l_1+m_1)}$$

$$\times \frac{[l_2+l_1-m-z]![2l_2-z]!}{[z]![l_1+l_2-l-z]![l_1+l_2+l+1-z]![l_2-m_2-z]!} \tag{3.17}$$

where the summation index z runs over all possible values such that the arguments of the symbol $[n]$ are non-negative. The Clebsch–Gordan coefficients do not depend on the parameters φ_i, ψ_i, $(i = 1, 2)$ and ϵ, i.e. CGC do not depend on the signature of the representation spaces. Exactly the same formula has been obtained in [20] where the particular case $\varphi_i = \psi_i = 0\,(i = 1, 2)$ was considered.

It is quite noticeable that this formula differs from the corresponding formula for $U_q(sl(2))$ Clebsch–Gordan coefficients only by the phase factor and by the definition of the symbol $[n]$.

Using methods similar to those described in [7], one can show that for the particular case $m = l$ the Clebsch–Gordan coefficients take the form

$$(l_1 m_1 \lambda_1, l_2 m_2 \lambda_2 | l l \lambda)_q$$

$$= (-1)^{\lambda_1(l-l_1-m_2)} (-1)^{(l_1-m_1)(l_2-m_2)+\frac{1}{2}(l_1-m_1)(l_1-m_1+1)}$$

$$\times q^{\frac{1}{4}(l_1+l_2-l)(l+l_2-l_1+1)} q^{-\frac{1}{2}(l_1-m_1)(l+1)}$$

$$\times \left(\frac{[2l+1]![l_2+m_2]![l_1+m_1]![l_1+l_2-l]!}{[l_1-m_1]![l_2-m_2]![l_2-l_1+l]![l_1-l_2+l]![l_1+l_2+l+1]!} \right)^{\frac{1}{2}} . \tag{3.18}$$

This formula has also been derived using a recursion relation in [19].

The analytical values of the simplest Clebsch–Gordan coefficients $(l_1\, m_1, 1\, m_2 | l\, m)_q$ are given in [20].

Note that

$$(l_1\, l_1\, \lambda_1,\; l_2\, l_2\, \lambda_2 |\; l_1 + l_2\;\; l_1 + l_2\; \lambda_1 + \lambda_2)_q = 1 \tag{3.19}$$

and that the Clebsch–Gordan coefficients $(l_1\, l_1\, \lambda_1,\, l_2\, m_2\, \lambda_2 | l\, l\, \lambda)_q$ are always positive. Therefore these Clebsch–Gordan coefficients satisfy the classical Condon-Shortley convention.

Finally note that the limit of the $U_q(osp(1|2))$ Clebsch–Gordan coefficients is the $osp(1|2)$ Clebsch–Gordan coefficients in the basis $e^l_m(\lambda)$, [20]

$$\lim_{q\to 1} (l_1 m_1 \lambda_1,\, l_2 m_2 \lambda_2 | l m \lambda)_q = (l_1 m_1 \lambda_1, l_2 m_2 \lambda_2 | l m \lambda)^c . \tag{3.20}$$

3.3. Properties of the Clebsch–Gordan coefficients

From equations (3.2) and (3.13), it follows that the Clebsch–Gordan coefficients satisfy the following pseudo-orthogonality relations:

$$\sum_{m_1 m_2} (-1)^{(l_1-m_1)(l_2-m_2)} (l_1 m_1 \lambda_1, l_2 m_2 \lambda_2 | l m \lambda)_q \, (l_1 m_1 \lambda_1, l_2 m_2 \lambda_2 | l' m' \lambda)_q$$

$$= (-1)^{(l-m)L} \delta_{ll'} \delta_{mm'} \tag{3.21}$$

$$\sum_{lm} (-1)^{(l-m)L} (l_1 m_1 \lambda_1, l_2 m_2 \lambda_2 | l m \lambda)_q \, (l_1 m_1' \lambda_1, l_2 m_2' \lambda_2 | l m \lambda)_q$$

$$= (-1)^{(l_1-m_1)(l_2-m_2)} \delta_{m_1 m_1'} \delta_{m_2 m_2'} \tag{3.22}$$

where $L = l_1 + l_2 + l$. Thus actually, the pseudo-orthogonality relations do not depend on the parameters φ_i, ψ_i, λ_i and ϵ.

Considering the action of operators $v_{\pm}^{\otimes}(1,2)$ on the defining relations for Clebsch–Gordan coefficients, one can derive the following recursion relations:

$$\sqrt{[l+m][l-m+1]}\gamma (l_1 m_1 \lambda_1, l_2 m_2 \lambda_2 | l m-1\, \lambda)_q$$

$$= q^{\frac{m_2}{2}} \sqrt{[l_1 - m_1][l_1 + m_1 + 1]}\gamma (l_1 m_1 + 1\, \lambda_1, l_2 m_2 \lambda_2 | l m\, \lambda)_q$$

$$+ (-1)^{l_1 - m_1 + \lambda_1} q^{\frac{-m_1}{2}} \sqrt{[l_2 - m_2][l_2 + m_2 + 1]}\gamma (l_1 m_1\, \lambda_1, l_2 m_2 + 1\, \lambda_2 | l m\, \lambda)_q \tag{3.23}$$

$$(-1)^{l_1+l_2+l+\lambda_1} \sqrt{[l-m][l+m+1]}\gamma (l_1 m_1 \lambda_1, l_2 m_2 \lambda_2 | l m+1\, \lambda)_q$$

$$= (-1)^{l_2 - m_2 + \lambda_1} q^{\frac{m_2}{2}} \sqrt{[l_1 + m_1][l_1 - m_1 + 1]}\gamma (l_1 m_1 - 1\, \lambda_1, l_2 m_2 \lambda_2 | l m\, \lambda)_q$$

$$+ q^{\frac{-m_1}{2}} \sqrt{[l_2 + m_2][l_2 - m_2 + 1]}\gamma (l_1 m_1\, \lambda_1, l_2 m_2 - 1\, \lambda_2 | l m\, \lambda)_q\,. \tag{3.24}$$

Such recursion relations were used in [19] to derive an analytical formula for Clebsch–Gordan coefficients equivalent to (3.18).

In the theory of the classical Racah–Wigner calculus, a very important role is played by the Clebsch–Gordan coefficient $(jm, jn|00)$ which defines an invariant metric in the representation space. In the case of the quantum superalgebra $U_q(osp(1|2)$, the corresponding coefficient has the form

$$C^{lq}_{mn}(\lambda) = (l m \lambda, l n \lambda | 000)_q = (-1)^{(\lambda)(l-m)} (-1)^{\frac{1}{2}(l-m)(l-m-1)} q^{\frac{m}{2}} \frac{1}{\sqrt{[2l+1]}} \delta_{m,-n}\,. \tag{3.25}$$

It also defines an invariant metric and satifies the properties:

$$C^{lq}_{mn}(\lambda) = (-1)^m C^{lq^{-1}}_{nm}(\lambda) \qquad C^{lq}_{mn}(\lambda) C^{lq^{-1}}_{pn}(\lambda) = \frac{\delta_{mp}}{[2l+1]}\,. \tag{3.26}$$

This invariant metric will be used to construct the symmetric 3-j symbols for the quantum superalgebra $U_q(osp(1|2))$. One can easily check that in the limit $q = 1$, the invariant metric becomes the invariant metric for the classical superalgebra $osp(1|2)$ [16].

3.4. Symmetries of the Clebsch–Gordan coefficients

Using techniques similar to those described in [7], or using the recursion relations (3.23), (3.24), one can prove that the Clebsch–Gordan coefficients possess the following symmetry properties:

$$(l_1\, m_1\, \lambda_1,\ l_2\, m_2\, \lambda_2 | l_3\, m_3\, \lambda_3)_q$$
$$= (-1)^{(l_1-m_1+\lambda_1)(l_3-m_2+\lambda_2)}\, (-1)^{(\lambda_1+\lambda_2)(l_1+l_2+l_3)+\lambda_1\lambda_2}\, (-1)^{\frac{1}{2}(l_1+l_2-l_3)(l_1+l_2-l_3+1)}$$
$$\times (l_2\, m_2\, \lambda_2,\ l_1\, m_1 \lambda_1 | l_3\, m_3\, \lambda_3)_{q^{-1}} \tag{3.27}$$

$$(l_1\, m_1\, \lambda_1,\ l_2\, m_2\, \lambda_2 | l_3\, m_3\, \lambda_3)_q$$
$$= (-1)^{\frac{1}{2}(l_2+m_2)(l_2+m_2-1)}\, (-1)^{\lambda_1(l_1+l_3-m_2)}$$
$$\times (-1)^{(\lambda_2+L)(l_1+l_2-m_3)} q^{-\frac{m_2}{2}} \left(\frac{[2l_3+1]}{[2l_1+1]}\right)^{\frac{1}{2}} (l_2\ -m_2\, \lambda_2,\ l_3\, m_3 \lambda_3 | l_1\, m_1\, \lambda_1)_{q^{-1}} \tag{3.28}$$

$$(l_1\, m_1\, \lambda_1,\ l_2\, m_2\, \lambda_2 | l_3\, m_3\, \lambda_3)_q$$
$$= (-1)^{\frac{1}{2}(l_1-m_1)(l_1-m_1-1)}\, (-1)^{\lambda_3(l_2+l_3-m_1)}$$
$$\times (-1)^{(\lambda_1+L)(l_1+l_3-m_2)} q^{\frac{m_1}{2}} \left(\frac{[2l_3+1]}{[2l_2+1]}\right)^{\frac{1}{2}} (l_3\, m_3\, \lambda_3,\ l_1\ -m_1 \lambda_1 | l_2\, m_2\, \lambda_2)_{q^{-1}}\,. \tag{3.29}$$

The Clebsch–Gordan coefficients also satisfy the 'mirror' symmetry

$$(l_1\, m_1\, \lambda_1,\ l_2\, m_2\, \lambda_2 | l_3\, m_3\, \lambda_3)_q$$
$$= (-1)^{\sum_{i=1}^{3} \frac{1}{2}(l_i-m_i)(l_i-m_i-1)} (l_1\ -m_1\, \lambda_1,\ l_2\ -m_2\, \lambda_2 | l_3\ -m_3\, \lambda_3)_{q^{-1}} \tag{3.30}$$

where $L = l_1 + l_2 + l_3$. All these symmetries have the same structure as the symmetries of Clebsch–Gordan coefficients for quantum algebra $U_q(su(2))$ [7], except that the phases are nonlinear in l_i, m_i and that they depend on the parities λ_i, $(i = 1, 2, 3)$.

Another similarity between both cases is the existence of Regge symmetry. However, in the case of the quantum superalgebra $U_q(osp(1|2))$, Regge symmetry is realized only under certain conditions. Assume that in the analytical formula (3.18) for Clebsch–Gordan coefficients the condition

$$l_1 + l_2 + m_1 + m_2 = 0 \qquad \mathrm{mod}(2) \tag{3.31}$$

is satisfied, and consider the following linear transformation on the arguments:

$$l_1' = \tfrac{1}{2}(l_1 + l_2 + m_1 + m_2) \qquad m_1' = \tfrac{1}{2}(l_1 - l_2 + m_1 - m_2) \tag{3.32}$$

$$l_2' = \tfrac{1}{2}(l_1 + l_2 - m_1 - m_2) \qquad m_2' = \tfrac{1}{2}(l_1 - l_2 - m_1 + m_2) \tag{3.33}$$

$$l_3' = l_3 \qquad m_3' = l_1 - l_2. \tag{3.34}$$

The superspins l_i', $i = 1, 2, 3$ satisfy the relation (3.8), the projections m_i' satisfy $m_1' + m_2' = m_3'$ and the condition (3.33) guarantees that l_i are integers. Thus, the transformation (3.34)–(3.36) is an admissible transformation on the superspins l_i and projections m_i in the tensor product $V^{l_1}(\lambda_1) \otimes V^{l_2}(\lambda_2)$. If we now substitute in the analytical formula (3.18) the values of l_i' and m_i', then the phase is invariant and all expressions in symbols $[n]$ either remain invariant or are exchanged pairwise so that the expression (3.18) remains unchanged. Therefore, the $U_q(osp(1|2))$ CGC satisfy Regge symmetry:

$$(l_1'm_1'\lambda_1, l_2'm_2'\lambda_2|l_3'm_3'\lambda_3)_q = (l_1m_1\lambda_1, l_2m_2\lambda_2|l_3m_3\lambda_3)_q . \tag{3.35}$$

Let us observe that the numerical value of the right-hand side of expression (3.18) is invariant under the transformation (3.32)–(3.33) even if the condition (3.31) is not satisfied. But in this case, the transformation introduces superspins that are half-odd-integers which do not correspond to irreducible representations of $U_q(osp(1|2))$. Thus, the analytical formula (3.18), obtained by application of the projection operator method, exhibits in a natural way the Regge symmetry of the coefficients.

4. Symmetric 3-j symbols for quantum superalgebra $U_q(osp(1|2))$

4.1. The parity-dependent sq3-jλ symbols

In analogy with the classical case of the $su(2)$ algebra, one can define $sq3$-$j\lambda$ symbols for the quantum superalgebra $U_q(osp(1|2))$ that possess good symmetry properties

$$\begin{pmatrix} l_1\lambda_1 & l_2\lambda_2 & l_3\lambda_3 \\ m_1 & m_2 & m_3 \end{pmatrix}_q = (-1)^{L\lambda_3}(-1)^{\frac{1}{2}(l_1+m_1)(l_1+m_1-1)}(-1)^{\frac{1}{2}(l_2-m_2)(l_2-m_2-1)}(-1)^{\frac{1}{2}(l_3+m_3)(l_3+m_3-1)} \times q^{\frac{1}{2}m_3-\frac{1}{6}(m_1-m_2)}C^{l_3q}_{m_3'm_3}(\lambda_3)(l_1m_1\lambda_1, l_2m_2\lambda_2|l_3m_3'\lambda_3)_q \tag{4.1}$$

where $C^{l_3q}_{m_3n_3}(\lambda_3)$ is the invariant metric defined by relation (3.28). Using the explicit form of the invariant metric, the definition of the symbol $sq3$-$j\lambda$ may be written

$$\begin{pmatrix} l_1\lambda_1 & l_2\lambda_2 & l_3\lambda_3 \\ m_1 & m_2 & m_3 \end{pmatrix}_q = (-1)^{(l_1+l_2-m_3)\lambda_3}(-1)^{\frac{1}{2}(l_1+m_1)(l_1+m_1-1)}(-1)^{\frac{1}{2}(l_2-m_2)(l_2-m_2-1)} \times \frac{q^{-\frac{1}{6}(m_1-m_2)}}{\sqrt{[2l_3+1]}}(l_1m_1\lambda_1, l_2m_2\lambda_2|l_3-m_3\lambda_3)_q . \tag{4.2}$$

The symbols $sq3$-$j\lambda$ satisfy the same constraints as Clebsch–Gordan coefficients, namely

$$|l_1 - l_2| \leqslant l_3 \leqslant l_1 + l_2 \tag{4.3}$$

$$m_1 + m_2 + m_3 = 0 \tag{4.4}$$

$$l_1 + l_2 + l_3 = \lambda_3 + \lambda_1 + \lambda_2 \quad \mathrm{mod}(2) . \tag{4.5}$$

Taking into account the symmetry properties of Clebsch–Gordan coefficients, one can show that $sq3$-$j\lambda$ symbols are invariant under even permutations of columns

$$\begin{pmatrix} l_1\lambda_1 & l_2\lambda_2 & l_3\lambda_3 \\ m_1 & m_2 & m_3 \end{pmatrix}_q = \begin{pmatrix} l_3\lambda_3 & l_1\lambda_1 & l_2\lambda_2 \\ m_3 & m_1 & m_2 \end{pmatrix}_q = \begin{pmatrix} l_2\lambda_2 & l_3\lambda_3 & l_1\lambda_1 \\ m_2 & m_3 & m_1 \end{pmatrix}_q . \tag{4.6}$$

Under an odd permutation of columns and the simultaneous change $q \to q^{-1}$, they are multiplied by a phase factor β

$$\begin{pmatrix} l_1\lambda_1 & l_2\lambda_2 & l_3\lambda_3 \\ m_1 & m_2 & m_3 \end{pmatrix}_q = \beta \begin{pmatrix} l_2\lambda_2 & l_1\lambda_1 & l_3\lambda_3 \\ m_2 & m_1 & m_3 \end{pmatrix}_{q^{-1}}$$

$$= \beta \begin{pmatrix} l_1\lambda_1 & l_3\lambda_3 & l_2\lambda_2 \\ m_1 & m_3 & m_2 \end{pmatrix}_{q^{-1}} = \beta \begin{pmatrix} l_3\lambda_3 & l_2\lambda_2 & l_1\lambda_1 \\ m_3 & m_2 & m_1 \end{pmatrix}_{q^{-1}} \tag{4.7}$$

where the phase is

$$\beta = (-1)^{\sum_{i=1}^{3} \frac{1}{2}(l_i - m_i - \lambda_i)(l_i - m_i - \lambda_i - 1)} . \tag{4.8}$$

Under 'mirror' symmetry, the $sq3\text{-}j\lambda$ symbols transform in a slightly different way

$$\begin{pmatrix} l_1\lambda_1 & l_2\lambda_2 & l_3\lambda_3 \\ -m_1 & -m_2 & -m_3 \end{pmatrix}_q = (-1)^{\sum_{i=1}^{3} \frac{1}{2}(l_i - m_i)(l_i - m_i - 1)} \begin{pmatrix} l_1\lambda_1 & l_2\lambda_2 & l_3\lambda_3 \\ m_1 & m_2 & m_3 \end{pmatrix}_{q^{-1}} . \tag{4.9}$$

Finally, if the condition

$$l_1 + l_2 + m_1 + m_2 = 0 \qquad \mathrm{mod}(2) \tag{4.10}$$

is satisfied, then the symbols satisfy Regge symmetry

$$\begin{pmatrix} l'_1\lambda_1 & l'_2\lambda_2 & l'_3\lambda_3 \\ m'_1 & m'_2 & m'_3 \end{pmatrix}_q = \begin{pmatrix} l_1\lambda_1 & l_2\lambda_2 & l_3\lambda_3 \\ m_1 & m_2 & m_3 \end{pmatrix}_q \tag{4.11}$$

where l'_i, m'_i, $i = 1, 2, 3$ are now of the form

$$l'_1 = \tfrac{1}{2}(l_1 + l_2 + m_1 + m_2) \qquad m'_1 = \tfrac{1}{2}(l_1 - l_2 + m_1 - m_2) \tag{4.12}$$

$$l'_2 = \tfrac{1}{2}(l_1 + l_2 - m_1 - m_2) \qquad m'_2 = \tfrac{1}{2}(l_1 - l_2 - m_1 + m_2) \tag{4.13}$$

$$l'_3 = l_3 \qquad m'_3 = l_2 - l_1 . \tag{4.14}$$

From the definition of the $sq3\text{-}j\lambda$ symbols, and from the pseudo-orthogonality relations for Clebsch–Gordan coefficients, it follows that these symbols satisfy modified pseudo-orthogonality relations:

$$\sum_{m_1 m_2} (-1)^{(l_1 - m_1)(l_2 - m_2)} q^{\frac{1}{3}(m_1 - m_2)} \begin{pmatrix} l_1\lambda_1 & l_2\lambda_2 & l_3\lambda_3 \\ m_1 & m_2 & m_3 \end{pmatrix}_q \begin{pmatrix} l_1\lambda_1 & l_2\lambda_2 & l'_3\lambda_3 \\ m_1 & m_2 & m'_3 \end{pmatrix}_q$$

$$= (-1)^{(l_3 - m_3)L} \frac{\delta_{l_3 l'_3} \delta_{m_3 m'_3}}{[2l_3 + 1]} \tag{4.15}$$

$$\sum_{l_3 m_3} (-1)^{(l_3 - m_3)L} [2l_3 + 1] \begin{pmatrix} l_1\lambda_1 & l_2\lambda_2 & l_3\lambda_3 \\ m_1 & m_2 & m_3 \end{pmatrix}_q \begin{pmatrix} l_1\lambda_1 & l_2\lambda_2 & l_3\lambda_3 \\ m'_1 & m'_2 & m_3 \end{pmatrix}_q$$

$$= (-1)^{(l_1 - m_1)(l_2 - m_2)} q^{-\frac{1}{3}(m_1 - m_2)} \delta_{m_1 m'_1} \delta_{m_2 m'_2} . \tag{4.16}$$

We also have the following relation between the $sq3\text{-}j\lambda$ symbols and the Clebsch–Gordan coefficients:

$$C^{l_3 q^{-1}}_{m_3 m'_3}(\lambda_3) \begin{pmatrix} l_1\lambda_1 & l_2\lambda_2 & l_3\lambda_3 \\ m_1 & m_2 & m'_3 \end{pmatrix}_q$$

$$= (-1)^{L\lambda_3} (-1)^{\frac{1}{2}(l_1 + m_1)(l_1 + m_1 - 1)} (-1)^{\frac{1}{2}(l_2 - m_2)(l_2 - m_2 - 1)}$$

$$\times \frac{q^{-\frac{1}{6}(m_1 - m_2)}}{[2l_3 + 1]} (l_1 m_1 \lambda_1, l_2 m_2 \lambda_2 | l_3 m_3 \lambda_3)^e_q . \tag{4.17}$$

In the limit $q \to 1$, the $sq3\text{-}j\lambda$ symbols become the $s3\text{-}j\lambda$ symbols for the superalgebra $osp(1|2)$ defined in [16].

The $sq3\text{-}j\lambda$ symbols have better symmetry properties than the Clebsch–Gordan coefficients, but they still depend on the parities λ_i, so these symbols are not real analogues of the $q3\text{-}j$ symbols for the quantum algebra $U_q(su(2))$. It was shown in [16] that in the case of the superalgebra $osp(1|2)$, the dependence on λ_i can be factored out, so that it was possible to define $s3\text{-}j$ symbols that do not depend on parities λ_i. In the next subsection, we will show that such a factorization is also possible in the case of the quantum superalgebra $U_q(osp(1|2))$, which gives rise to the possibility of defining parity-independent $sq3\text{-}j$ symbols.

4.2. Parity-independent sq3-j symbols

Parity-independent $sq3\text{-}j$ symbols are defined in the following way:

$$\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}_q = (-1)^{\sum_{\rm circ}(l_i - m_i)(l_{i+1}+\lambda_{i+1})} \begin{pmatrix} l_1\lambda_1 & l_2\lambda_2 & l_3\lambda_3 \\ m_1 & m_2 & m_3 \end{pmatrix}_q \tag{4.18}$$

with the short notation

$$\sum_{\rm circ} x_i y_{i+1} = x_1 y_2 + x_2 y_3 + x_3 y_1 \,. \tag{4.19}$$

Using the relation (4.2), the $sq3\text{-}j$ symbols may be expressed in terms of Clebsch–Gordan coefficients

$$\begin{aligned} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}_q &= (-1)^{\lambda_1(l_1+l_3-m_2)} (-1)^{l_1(l_2+l_3-m_1)} (-1)^{l_2(l_1+l_2-m_3)} (-1)^{L(l_1+l_2-m_3)} \\ &\quad \times (-1)^{\frac{1}{2}(l_1+m_1)(l_1+m_1-1)} (-1)^{\frac{1}{2}(l_2-m_2)(l_2-m_2-1)} \frac{q^{-\frac{1}{6}(m_1-m_2)}}{\sqrt{[2l_3+1]}} \\ &\quad \times (l_1\, m_1\, \lambda_1, l_2\, m_2\, \lambda_2 | l_3\, -m_3\, \lambda_3)_q \,. \end{aligned} \tag{4.20}$$

The symbols satisfy constraints (4.3) and (4.4). Using the analytical formula for Clebsch–Gordan coefficients, one can easily check that the $sq3\text{-}j$ symbols do not depend on the parities λ_i.

The symmetry properties satisfied by the parity-independent $sq3\text{-}j$ symbols are similar to the symmetry properties of $sq3\text{-}j\lambda$ symbols:

$$\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}_q = \begin{pmatrix} l_3 & l_1 & l_2 \\ m_3 & m_1 & m_2 \end{pmatrix}_q = \begin{pmatrix} l_2 & l_3 & l_1 \\ m_2 & m_3 & m_1 \end{pmatrix}_q \tag{4.21}$$

$$\begin{aligned} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}_q &= \alpha \begin{pmatrix} l_2 & l_1 & l_3 \\ m_2 & m_1 & m_3 \end{pmatrix}_{q^{-1}} = \alpha \begin{pmatrix} l_1 & l_3 & l_2 \\ m_1 & m_3 & m_2 \end{pmatrix}_{q^{-1}} \\ &= \alpha \begin{pmatrix} l_3 & l_2 & l_1 \\ m_3 & m_2 & m_1 \end{pmatrix}_{q^{-1}} , \end{aligned} \tag{4.22}$$

where the phase α is

$$\alpha = (-1)^{\sum_{i=1}^{3} \frac{1}{2}(l_i - m_i)(l_i - m_i - 1)} (-1)^{\sum_{i=1}^{3} l_i m_i} \,. \tag{4.23}$$

Under 'mirror' symmetry, the $sq3\text{-}j$ symbols transform in the same way as $sq3\text{-}j\lambda$ symbols

$$\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}_q = (-1)^{\sum_{i=1}^{3} \frac{1}{2}(l_i - m_i)(l_i - m_i - 1)} \begin{pmatrix} l_1 & l_2 & l_3 \\ -m_1 & -m_2 & -m_3 \end{pmatrix}_{q^{-1}} . \tag{4.24}$$

In addition, there exists a conditional Regge symmetry, but in this case the symbol is not invariant, it is changed by a phase factor

$$\begin{pmatrix} l'_1 & l'_2 & l'_3 \\ m'_1 & m'_2 & m'_3 \end{pmatrix}_q = (-1)^{(l_1+l_3+m_2)(l_2+(1/2)(l_1+l_2+m_3)} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}_q \tag{4.25}$$

where $l'_i, m'_i, i = 1, 2, 3$ are given by formulae (4.12)–(4.14) and l_i and m_i satisfy the condition (4.10) which guarantees that the phase is real.

The $sq3\text{-}j$ symbols satisfy pseudo-orthogonality relations identical to those of $sq3\text{-}j\lambda$ symbols equations (4.15)–(4.16). From the parity-independent $sq3\text{-}j$ symbol, one can define an invariant metric that is independent of λ

$$C^{lq}_{mn} = q^{\frac{1}{3}m} \begin{pmatrix} l & l & 0 \\ m & n & 0 \end{pmatrix}_q = (-1)^{l(l-m)} (-1)^{\frac{1}{2}(l-m)(l-m-1)} \frac{q^{\frac{m}{2}}}{\sqrt{[2l+1]}} \delta_{m,-n} \,. \tag{4.26}$$

It is related to the invariant metric $C^{lq}_{mn}(\lambda)$ defined by relation (3.29) in the following way:

$$C^{lq}_{mn} = (-1)^{(l-m)(\lambda+l)} C^{lq}_{mn}(\lambda) \,. \tag{4.27}$$

In the limit $q \rightarrow 1$, the $sq3\text{-}j$ symbols become $s3\text{-}j$ symbols for the superalgebra $osp(1|2)$ [16].

5. Conclusion

The quantum superalgebra $U_q(osp(1|2))$ can be considered as the quantum analogue of $osp(1|2)$ superalgebra. The grade-star represetations of the quantum superalgebra $U_q(osp(1|2))$ are superanalogues of Hermitian representations of $U_q(su(2))$ quantum algebra.

In this paper, it has been shown that the irreducible representations of the quantum superalgebra $U_q(osp(1|2))$ have the same structure as those of the non-deformed superalgebra $osp(1|2)$. In particular, Clebsch–Gordan coefficients have been defined such that they satisfy the same symmetry properties and pseudo-orthogonality relations as in the non-deformed case. Moreover, after factorization of the parity dependence, we have defined symmetric $sq3\text{-}j$ symbols which are, at the same time, quantum deformations of the $s3\text{-}j$ symbols for the superalgebra $osp(1|2)$ and supersymmetric analogues of the $q3\text{-}j$ symbols for the quantum algebra $U_q(su(2))$.

In a forthcoming publication, this analysis of the Racah–Wigner calculus for the $U_q(osp(1|2))$ quantum superalgebra will be continued with the definition and analysis of $sq6\text{-}j$ symbols.

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